



THE CMBR SPECTRUM

A Theoretical Introduction

ALBERT STEBBINS

*NASA/Fermilab Astrophysics Group
Box 500, Batavia, IL 60510, USA*

1. Introduction

The Cosmic Microwave Background Radiation (CMBR) provides a strong observational foundation for the standard cosmological scenario, the Big Bang theory. It is difficult to understand how to produce a 2.7°K blackbody spectrum except in the context of the Big Bang scenario. The near blackbody spectrum of the CMBR along with its near isotropy provides compelling evidence for a period of fairly quiescent Friedman-Robertson-Walker expansion for many expansion time before recombination. The past decade has seen huge advances in the measurement of the CMBR, with COBE's definitive discovery of anisotropies and measurement of a near perfect blackbody spectrum. The small deviations from isotropy have and will continue to tell us a great deal about the inhomogeneities in our universe, and small deviations from a blackbody spectrum can also tell us about the energetics in our universe. Such deviations have already been discovered in the direction of clusters of galaxies, although the mean CMBR spectrum is, so far, indistinguishable from a blackbody spectrum.

Here we give a introduction to the observed spectrum of the CMBR and discuss what can be learned about it. Particular attention will be given to how Compton scattering can distort the spectrum of the CMBR. This is left toward the end though. Unfortunately the author has no expertise in the area of how these measurements are made but Smoot has covered this area in his lectures. An incomplete bibliography of relevant papers is also provided. Some old but still highly useful reviews of the physics behind the spectra are by Danese and De Zotti[20], and Sunyaev and Zel'dovich[74]. Theoretically not much has changed in this field in over 25 years. Much of the interesting work was done by Zel'dovich and Sunyaev in 1969[83].

2. Executive Summary

The universe today is fairly diffuse and cold, however the universe is observed to be expanding, and in the past we may deduce that the universe was more dense and because of $p dV$ work the matter in the universe would also have been hotter. Extrapolating the expansion back to very early epochs the universe would have been very hot and very dense and the universe must have been expanding very rapidly in order to have grown as large as it is observed to be. Hence the Hot Big Bang. When the matter in the universe is hot and dense the thermal equilibration time becomes very short. Thus we expect everything to rapidly approach thermal equilibrium and we therefore expect the photons in the universe to have a thermal (blackbody) spectrum at early times. It is easily shown that expansion of the universe (or traversal through any gravitational field) leaves a blackbody spectrum a blackbody spectrum although the temperature may change. This temperature change is known as the redshift and can sometimes be thought of as either a Doppler shift or a gravitational redshift. Formally speaking the two may be thought of as the same phenomena and there is often no physical sense in trying to separate them.

Thus as a first approximation we expect the photons in the universe to have a blackbody spectrum. The fact that the cosmic microwave background radiation (CMBR) has nearly a blackbody spectrum is strong evidence for the Hot Big Bang hypothesis. There is simply not enough matter around today to thermalize so many photons (there are $\gtrsim 10^9$ photons for every atom) and in any case most of the matter in our universe is much hotter than 3 K. The reason we might expect a deviation from blackbody is because some of the matter in the universe has gone out of thermal equilibrium with the photons and may either heat or cool the photons. This can be done by non-equilibrium scattering or absorption of existing photons or by non-equilibrium emission of new photons. Clearly most of the matter in the universe is not today in thermal equilibrium with the CMBR and the spectrum offers us a probe of this. However there are so many more CMBR photons in the universe than there are protons or electrons that it is difficult for the matter to significantly distort the spectrum of the CMBR. Thus the fact that the observed CMBR spectrum is so close to a blackbody should come as no surprise.

3. Measures of Temperature

The *brightness* or *specific intensity* of light, I_ν , is defined as the incident energy per unit area, per unit solid angle, per unit frequency, per unit time.

It may be written

$$I_\nu = \frac{2h\nu^3}{c^2} n_\gamma \quad (1)$$

where ν is the frequency and $n_\gamma(\nu)$ is the quantum-mechanical occupation number, i.e. the number of photons (in each polarization state) per unit phase space volume measured in units of h^3 . Here h is Planck's constant, and we assume the light is not (linearly) polarized so that there are an equal number of photons in each polarization state. A *blackbody* or *Planck spectrum* has

$$n_\gamma^{\text{BB}} = \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad (2)$$

where T is the temperature. The high-frequency ($h\nu \gg kT$) limit of the Planck spectrum is known as *Wien's law*:

$$I_\nu^{\text{W}} = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right) \quad (3)$$

while the low frequency ($h\nu \ll kT$) limit of the Planck spectrum is known as the *Rayleigh-Jeans law*:

$$I_\nu^{\text{RJ}} = \frac{2\nu^2 kT}{c^2} . \quad (4)$$

Note that the intensity is proportional to the temperature in this case. One may invert the Planck spectrum and characterize the intensity by the *thermodynamic temperature* or *brightness temperature*:

$$T_{\text{b}} = \frac{h\nu}{k \ln \left| 1 + \frac{2h\nu^3}{c^2 I_\nu} \right|} \quad (5)$$

Occasionally radio astronomers may define the brightness temperature by it's Rayleigh Jeans limit:

$$T_{\text{b}}^{\text{RJ}} = \frac{c^2 I_\nu}{k 2\nu^2} . \quad (6)$$

In the radio region this is an excellent approximation to the thermodynamic temperature and is simply related to the intensity, and is therefore closer to what is actually measured.

Here we are interested in small deviations from a blackbody spectrum, i.e. we have some temperature T_γ which is a good fit to T_{b} at many frequencies, and want to express the actual spectrum in terms of small deviations

from a blackbody with this temperature, in particular in terms of the deviations in intensity from the blackbody spectrum, ΔI_ν . For small deviations the deviation in brightness temperature is

$$\Delta T_b \equiv T_b - T_\gamma = \frac{(e^x - 1)^2}{x^2 e^x} \frac{c^2 \Delta I_\nu}{k 2\nu^2} \quad x \equiv \frac{h\nu}{kT_\gamma} . \quad (7)$$

Experimentally it is often easier to measure differences rather than absolute numbers: differences in intensity in different directions on the sky, or between the sky and internal calibrators. Note that in the Wien region differences in brightness temperature are greater than in the Rayleigh-Jeans region for the same difference in intensity. In what follows we will tend to plot spectral distortions in terms of differences in brightness temperature versus the dimensionless frequency, x . These “derived” quantities are probably more appealing to a theorist than an observer since they are further removed from what is actually measured.

4. Measured Mean Spectrum of the CMBR

Over the years there have been many measurements of the CMBR. There have been many claims that the spectrum deviated significantly from a blackbody, especially in the Wien region, however recent measurements with FIRAS (Far-InfraRed Absolute Spectrophotometer) on the COBE satellite has shown definitively that the spectrum is very close to a blackbody[48]. Contemporary and quite competitive with with the first FIRAS measurements was a rocket experiment [34] which also found a blackbody. The most recent FIRAS results have appeared in ref [30] which are plotted in figs 1 & 2. In the 2nd figure the we have converted to a more theoretical representation by plotting versus $x = \frac{h\nu}{kT_\gamma}$ and converting to fractional changes in temperature. The reason that this transformation is useful is that we can predict the shape of the deviations from blackbody in terms of x , which we cannot in terms of ν since we have no *a priori* knowledge of T_γ . To transform to an x variable one must decide on a fiducial temperature. We have used the best-fit temperature, $T_\gamma = 2.728\text{K}$, taken from the most recent results of FIRAS[30]: $T_\gamma = 2.728 \pm 0.002\text{K}$. Note that the uncertainty in T_γ is which is larger than the error bars on most of the individual points in the plots. The reason that the uncertainty in $T_b - T_\gamma$ can be smaller than the uncertainty in T_γ is that the experiment measures the difference in brightness between a blackbody and the sky. The $\pm 0.002\text{K}$ represents the uncertainty in the temperature of the reference blackbody, while the accuracy to which this reference is thought to be a blackbody is much better than this. Note that when fitting for a distortion from a blackbody one must fit for both the amplitude of the distortion and for T_γ simultaneously.

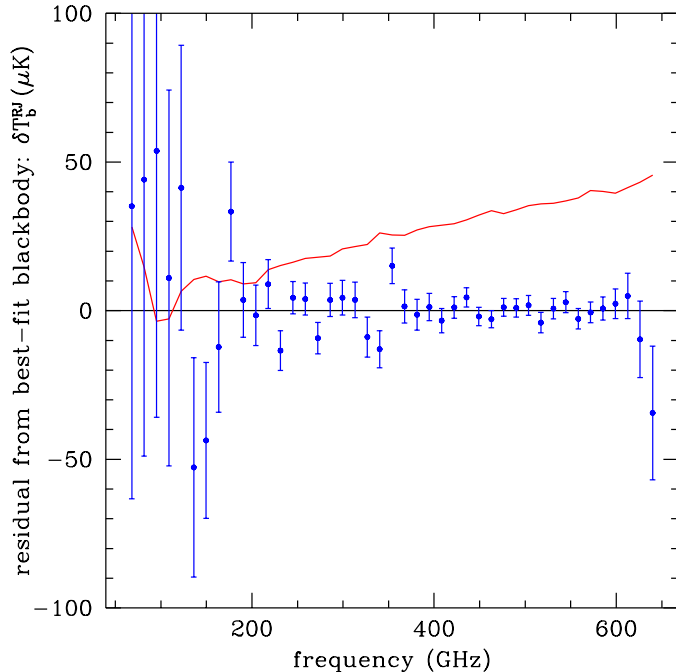


Figure 1. Plotted are the residuals in Rayleigh-Jeans temperature from the best fit blackbody as a function of frequency as stated by Fixsen *et al.* (1996). The error bars are $1\text{-}\sigma$. The solid line is the subtracted Galaxy model at the Galactic poles. We see that these measurements are running up against a fundamental limitation of Galactic contamination.

While FIRAS certainly revolutionized the field, and does make obsolete most other short wavelength measurements of the CMBR spectrum, it only looked at the frequency range $68\text{--}640\text{GHz}$. The bolometric techniques used by FIRAS only work at high frequencies and therefore the spectrum at low frequency was not touched by FIRAS. There has been ongoing measurements of the absolute CMBR flux in the Rayleigh-Jeans region for 30 years and we list some results from the last 15 years in Table 1. One can see that measurement did not stop after COBE. As we shall see some of the spectral distortions we are looking for are most visible in the Rayleigh-Jeans regime. We have selected some of the most sensitive of measurements to plot in fig 3. The uncertainties vary widely with frequency and are orders of magnitude larger than those of FIRAS. Several authors have noted that these low frequency measurements tend to indicate a temperature lower than that obtained at higher frequencies[62, 7], suggesting that there may

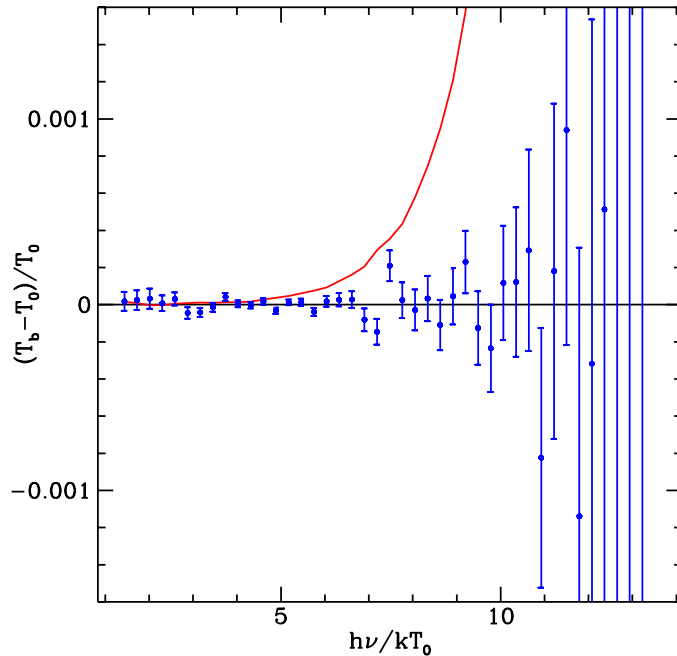


Figure 2. Same as fig 1 except here we plot the fractional deviation in brightness temperature vs. the dimensionless frequency $x = \frac{h\nu}{kT}$. To do this we have used the best-fit photon temperature $T_\gamma = 2.728$ K. Plotting things in this way accentuates the deviations at high frequencies.

be a deviation from a blackbody spectrum at low frequencies.

Measurement of the absolute CMBR spectrum, at the present level of sensitivity, face significant problems of Galactic contamination at both long and short wavelengths. Synchrotron radiation contaminates the long-wavelength spectrum while the short wavelength region is contaminated by dust emission. Since we cannot expect to observe the CMBR from outside of the Galaxy this is a fundamental limitation. Many of the results plotted here include significant corrections for this contamination. While there is a limit to how well one can subtract off the Galaxy, we can look forward to improvements in Galaxy modeling using results from anisotropy experiments which will have increasingly better sensitivity, sky coverage, and angular resolution. While anisotropy experiments cannot generally make absolute measurements of intensity, they can help to map out the Galaxy.

TABLE 1. Listed are measurements, made over the past 15 years, of the absolute CMBR brightness temperature at a variety of wavelengths. The results often include significant corrections for Galactic emission. Millimeter wavelengths are omitted as they have been superseded by results of FIRAS (> 68 GHz - see Fixsen *et al.* 1996). The ADS code given refers to the paper where these results are presented or reviewed and may be used to find the papers and abstracts online at the NASA Astrophysics Data System and mirror sites: *adsabs.harvard.edu*, *cdsads.u-strasbg.fr*, *d01.mtk.nao.ac.jp*. These codes are of the form *year journal volume page*.

Frequency (GHz)	Wavelength (cm)	T_{CMBR} (Kelvin)	1st Author	ADS Bibliographic Code
1.47	20.4	$2.26^{+0.19}_{-0.19}$	Bensadoun	1993ApJ...409....1B
90.	0.22	$2.60^{+0.09}_{-0.09}$	Bersanelli	1989ApJ...339..632B
2.0	15.	$2.55^{+0.14}_{-0.14}$	"	1994ApJ...424..517B
3.7	8.1	$2.59^{+0.13}_{-0.13}$	De Amici	1988ApJ...329..556D
3.8	7.9	$2.64^{+0.07}_{-0.07}$	"	1990ApJ...359..219D
3.8	7.9	$2.64^{+0.06}_{-0.06}$	"	1991ApJ...381..341D
25.	1.2	$2.783^{+0.025}_{-0.025}$	Johnson	1987ApJ...313L...1J
7.5	4.0	$2.60^{+0.07}_{-0.07}$	Kogut	1990ApJ...355..102K
1.410	21.26	$2.11^{+0.38}_{-0.38}$	Levin	1988ApJ...334...14L
7.5	4.	$2.64^{+0.06}_{-0.06}$	"	1992ApJ...396....3L
4.75	6.3	$2.70^{+0.07}_{-0.07}$	Mandolesi	1986ApJ...310..561M
2.5	12.	$2.79^{+0.15}_{-0.15}$	Sironi	1986ApJ...311..418S
0.600	50.	$3.0^{+1.2}_{-1.2}$	"	1990ApJ...357..301S
2.5	12.	$2.50^{+0.34}_{-0.34}$	"	1991ApJ...378..550S
0.82	36.6	$2.7^{+1.6}_{-1.6}$	"	"
2.5	12.0	$2.78^{+0.3}_{-0.3}$	Smoot	1987ApJ...317L..45S
33.0	0.909	$2.81^{+0.2}_{-0.2}$	"	"
1.41	21.2	$2.22^{+0.55}_{-0.55}$	"	"
3.66	8.2	$2.59^{+0.14}_{-0.14}$	"	"
10.	3.0	$2.61^{+0.06}_{-0.06}$	"	"
90.	0.33	$2.60^{+0.10}_{-0.10}$	"	"
1.4	21.	$2.65^{+0.33}_{-0.30}$	Staggs	1993PhDT.....6S
10.7	2.80	$2.730^{+0.014}_{-0.014}$	"	1996ApJ...473L...1S
90.	0.33	$2.57^{+0.12}_{-0.12}$	Witebsky	1986ApJ...310..145W

5. Spectral Distortions of the CMBR

While one should not be surprised that the CMBR has close to a blackbody spectrum, there are various mechanisms which should cause deviations from a thermal spectrum. Now we discuss a few of them.

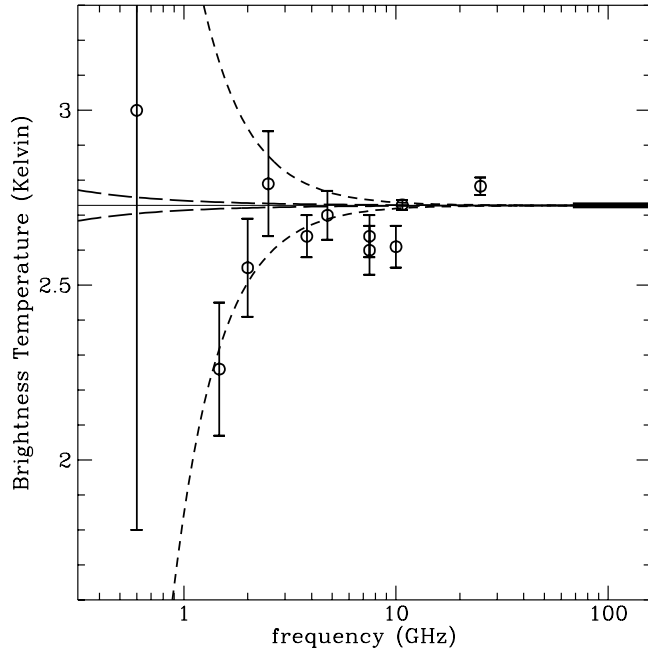


Figure 3. Plotted are a selection of the low frequency measurements of the CMBR brightness temperature listed in table 1. From left to right the points are from Sironi *et al.* (1990), Bensadoun *et al.* (1993), Bersanelli *et al.* (1994), Sironi & Bonelli (1986), De Amici *et al.* (1991), Mandolesi *et al.* (1986), Kogut *et al.* (1990), Levin *et al.* (1992), Smoot *et al.* (1987), Staggs (1996), Johnson & Wilkinson (1987) and were chosen because of the small errorbars. The black band at the right indicates the FIRAS data (Fixsen *et al.* 1996), while the horizontal straight line represents a temperature 2.728 K given by the FIRAS best fit blackbody spectrum. The long-dashed represents a chemical potential distortions with amplitude $\mu = \pm 9 \times 10^{-5}$ while the solid line gives free-free distortions with amplitude $Y_{\text{ff}} = \pm 10^{-4}$. These are both idealized curves and one may expect (model dependent) corrections long-ward of 10 GHz (see Burigana, De Zotti, and Danese 1995).

5.1. ANISOTROPIES

The most common way in which the CMBR spectral distortion occurs is when the photons have a blackbody spectrum in each direction but the temperatures characterizing these spectra are different in different directions. This direction dependent temperature difference is called anisotropy. The anisotropy can either be caused by Doppler/gravitational effect or because the gas emitting the photons really did have different temperatures.

The first anisotropy discovered was the dipole anisotropy, i.e. the temperature varies like the cosine of the angle from some point on the celestial

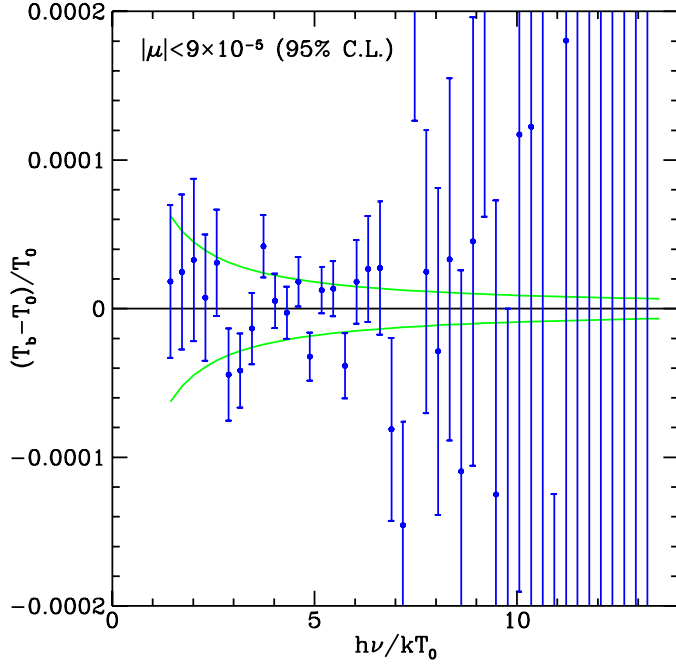


Figure 4. Superimposed on the FIRAS data of fig 2 is the largest chemical potential distortion allowed by the data (Fixsen *et al.* 1996): $\mu = \pm 9 \times 10^{-5}$. The falling positive curve is the far more plausible positive chemical potential distortion and the negative rising curve is a negative chemical potential distortion.

sphere. It is usually attributed to the Sun moving at 371km/s. Note that to a first approximation, when averaging over the sky, the dipole does not contribute to the mean spectrum of the CMBR.

One way to check that measured anisotropies are what they are supposed to be is to measure the spectrum. For a small anisotropy the change in flux from the mean spectrum should be proportional to the derivative of the flux of a blackbody with respect to temperature. FIRAS has done just that for the dipole[29, 30] and found just what was expected. Most modern anisotropy experiments use many frequency channels in order to check the spectrum of the anisotropy, or more specifically to be able to subtract off contamination of the measurements by other effects than the anisotropy.

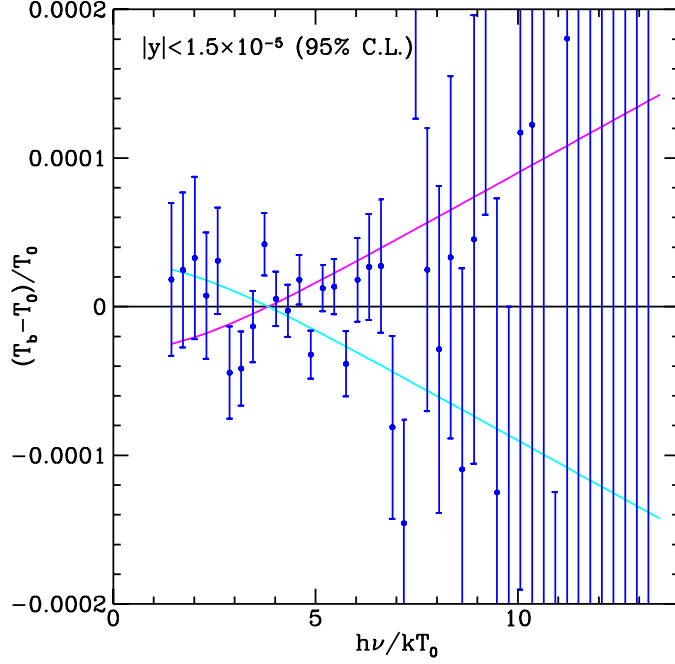


Figure 5. Superimposed on the FIRAS data of fig 2 is the largest y -distortion allowed by the data (Fixsen *et al.* 1996): $y = \pm 1.5 \times 10^{-5}$. The rising curve is the more plausible positive y -distortion and the falling curve is a negative y -distortion.

5.2. CHEMICAL POTENTIAL DISTORTIONS

There are three processes which are important from thermalizing the CMBR spectrum in the early universe: *Compton scattering*, *double Compton scattering*, and *free-free scattering* (also known as *bremsstrahlung*). Compton scattering is a much more rapid process but since it conserves the number of photons so it can only thermalize the energy distribution of the photons and not the number of photons. All of these processes become more efficient as one goes to earlier and earlier epochs and eventually photon non-conserving processes start to become important.

There is an epoch between $z = 10^5$ and $z = 2 \times 10^6$ during which Compton scattering is efficient in thermalizing the energy distribution while other processes are not capable of thermalizing the photon number. During this epoch, if the energy-to-photon ratio is perturbed from that required for a blackbody spectrum, the spectrum will instead approach a Bose-Einstein

distribution

$$n^{\text{BE}} = \frac{1}{\exp\left(\frac{h\nu}{kT_\gamma} + \mu\right) - 1} \quad (8)$$

where T_γ and μ (the *dimensionless chemical potential*) are determined by the total energy available and the total number of photons available. If one starts out with a thermal distribution of photons at temperature T_γ and injects a fractional increase in the energy density, $\frac{\Delta U}{U}$, without significantly increasing the number of photons one obtains

$$T_b \approx T_\gamma \left(1 - \mu \left[0.456 - \frac{1}{x}\right]\right) \quad \frac{\Delta U}{U} = 0.714\mu \quad \mu \ll 1 \quad (9)$$

Since one must fit the observations to both T_γ and μ one really can only measure the $\frac{1}{x}$ term. Double Compton scattering and free-free scattering become increasingly more efficient at lower frequencies and there are usually corrections to this formula at small frequencies $x \ll 1$ [9, 10]. These corrections are not liable to be important for FIRAS measurements.

This distortion to the spectrum is greatest at small x , however the FIRAS measurements at high frequencies are so accurate that they yield much better constraints on μ than does the low frequency experiments. Comparing with the FIRAS data one finds $|\mu| < 9 \times 10^{-5}$ at the 2σ level[30]. We compare the maximal allowed distortion to the low & high frequency data in figs 3&4, respectively.

Thus we find the extremely stringent constraint at a very early epoch

$$\frac{\Delta U}{U} < 6 \times 10^{-5} \quad 10^5 < z < 2 \times 10^6 \quad (10)$$

Of course this is not to say that one expects large energy injection at these epochs.

Note that for $z > 10^6$ the CMBR spectrum is not telling us much about the energetics of the universe. However one can use Big Bang Nucleosynthesis to probe the total energy of the universe up to $z \sim 10^{10}$.

5.3. Y DISTORTIONS

If energy is injected into the universe after $z \sim 10^4$ Compton scattering is unable to thermalize the distribution. The fact that we observe very little deviation from a blackbody spectrum tells us that not much energy could have been injected compared with the thermal energy of the CMBR. If a small amount of energy is injected then one may solve for the linear perturbation from a blackbody spectrum under the action of Compton

scattering as was done by Zel'dovich and Sunyaev[83]. One finds that the perturbation in the photon occupation number is

$$\Delta n = y \frac{x e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) \quad (11)$$

where the “ y -parameter” is

$$y = \int dt \sigma_T c N_e \frac{k(T_e - T_\gamma)}{m_e c^2}, \quad (12)$$

N_e is the number density of free-electrons, and σ_T is the Thomson cross-section. If one could manage to cool gas below the radiation temperature one could produce a distortion with negative y , but typically this distortion is produced by ionized gas which is much hotter than the photons. In the early universe when the density of electrons is large even a small heating of the gas over the photon temperature may lead to a significant distortion. This distortion is generally referred to as a y -distortion when applied to the mean CMBR spectrum, but is usually called the Sunyaev-Zel'dovich distortion when referring to an anisotropy in the spectrum because there is more or less hot gas in one direction than another. Large amounts of hot ionized gas exist in clusters of galaxies and the “S-Z effect” has been observed in the directions of several clusters. There is no evidence for a y of the mean CMBR spectrum, although with sensitive enough measurements we should see the hot gas we know is out there. Fixsen *et al.*[30] have placed a limit of $|y| < 1.5 \times 10^{-5}$ from the FIRAS data. We compare the maximal distortions to the FIRAS data in fig 5. Note that a positive y -distortion produces a negative change in T_b at low frequencies and positive change in T_b at high frequencies, just what one expect if one was heating a fixed number of photons. This negative ΔT_b is sometimes called the “S-Z decrement”, for the S-Z effect was first looked for at radio wavelengths.

5.4. DISTORTION FROM FREE-FREE

Another important process for the CMBR spectrum is free-free scattering, which is the scattering of a free electron off of a charged nucleon either emitting or absorbing a photon; in most cases of interest emitting rather than absorbing. This is the same process which produces the X-rays observed from hot cluster gas operating at microwave and radio frequencies. The effect of free-free scattering on the CMBR spectrum is mostly likely to be seen at the very longest wavelengths measured. In this Rayleigh-Jeans limit the distortion it produces may approximated by[3]

$$\frac{\Delta T_b}{T_\gamma} \approx \frac{Y_{\text{ff}}}{x^2} \quad Y_{\text{ff}} = \int dt \frac{T_e - T_\gamma}{T_e} \kappa dt \quad (13)$$

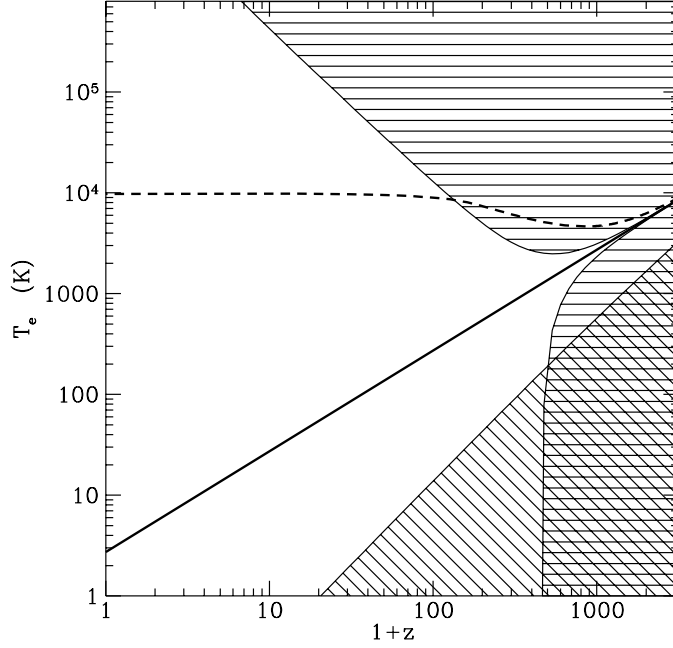


Figure 6. Plotted is the constraint on the temperature of a fully ionized universe as a function of redshift. The horizontally hatched region is excluded since $|y| < 1.5 \times 10^{-5}$ while the diagonally hatched region would be excluded if $Y_{\text{ff}} < 10^{-4}$. The solid line indicates where the gas temperature equals the photon temperature and the dashed line gives the temperature as a function of redshift for a model where the gas is ionized by very massive stars (VMOs - see Stebbins & Silk 1986). The cosmological parameters used are $H_0 = 65 \text{ km/sec/Mpc}$, $\Omega_0 = 0.4$ and $\Omega_b = 0.10$.

where

$$\kappa \equiv \frac{8\pi e^6 h^2 N_e^2 g}{3m_e (kT_\gamma)^3 \sqrt{6\pi m_e kT_e}} \quad (14)$$

and the Gaunt factor, $g \sim 2$ in most cases of interest. Note the $T_e^{-\frac{1}{2}}$ factor in κ which means that the effect is suppressed for higher temperature gas. The $1/x^2$ dependence on this distortion means that it is the low frequency measurements which will constrain its amplitude. In fig 3 we plot free-free distortions with $Y_{\text{ff}} = \pm 10^{-4}$. We see that this size distortion is close to what is being constrained by these measurements, although no proper statistical analysis has been done. This size free-free distortion would produce no significant effect in the FIRAS data, although if one went far enough into the Wien tail one would find large distortions from free-free emission.

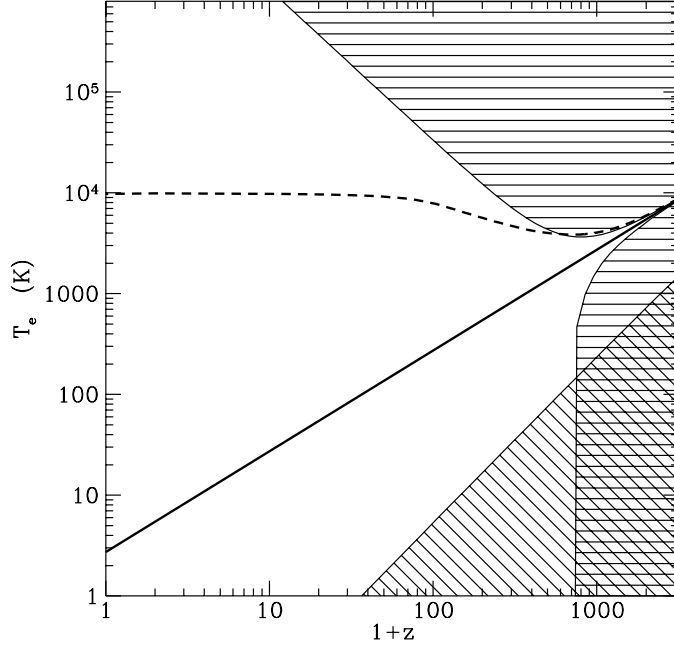


Figure 7. The same as the previous figure except with $\Omega_0 = 1$ and $\Omega_b = 0.06$. The constraints are less severe for for larger Ω_0 and smaller Ω_b .

In figs 6&7 we have used constraints on y and Y_{ff} to put constraints on the temperature and epoch of a reionized universe, assuming presently favored cosmological parameters. We see that it really the y -distortion which is most important, the free-distortion only being detectable on the off chance that the gas was ionized and cold. Even though the limits on y are quite small we see that there is not too much of a constraint of ionization after $z \sim 100$. The constraints could be made stronger if one assumes a larger baryon density or a smaller total density.

6. Physical Processes

Now we will take a closer look at the physical processes which could cause a distortion of the spectrum. Here we will only discuss Compton scattering although free-free emission and double-Compton deserve an equally thorough treatment.

6.1. COMPTON SCATTERING

6.1.1. Collisional Boltzmann Equation

One may describe the state of the primeval gas of photons and electrons in terms of the density of particles in phase space, i.e. momentum and position space. Here we are not interested in the polarization state of electrons and photons so we average over the two polarization states ¹ It is convenient to measure the phase space density in units of $h = 2\pi\hbar$ which gives the quantum mechanical occupation number, n_γ and n_e for photons and electrons, respectively. The evolution of n_γ can be described by the collisional Boltzmann equation which has the form

$$\frac{Dn_\gamma(\mathbf{p}_\gamma)}{Dt} = C(\mathbf{p}_\gamma) \quad (15)$$

where $C(\mathbf{p}_\gamma)$ is the scattering term which describes the interactions with other particles. Here $\frac{D}{Dt}$ is a convective derivative along the photon's trajectory in phase space, while the right-hand-side gives the collision integral. If there were no collisions then the Boltzmann equation states that the occupation number remains constant along photon trajectories. ² Included in this convective derivative are all the effects of gravity on the photons, which include many of the effects which produce anisotropy. We will not discuss these effects further as they are covered extensively in Bunn's lectures.

The collision integral for Compton scattering of unpolarized particles after averaging over the polarization state of scattered particles is of the form³

$$\begin{aligned} C^C(\mathbf{p}_\gamma) = & \frac{2}{(2\pi\hbar)^3} \int d^3\mathbf{p}_e \int d^2\hat{\mathbf{n}}' c(1 - \hat{\mathbf{n}} \cdot \vec{\beta}) \frac{d^2\sigma}{d^2\hat{\mathbf{n}}'} \\ & \times \left[(1 - n_e(\mathbf{p}_e)) n_e(\mathbf{p}'_e) (1 + n_\gamma(\mathbf{p}_\gamma)) n_\gamma(\mathbf{p}'_\gamma) \right. \\ & \left. - (1 - n_e(\mathbf{p}'_e)) n_e(\mathbf{p}_e) (1 + n_\gamma(\mathbf{p}'_\gamma)) n_\gamma(\mathbf{p}_\gamma) \right] \quad (16) \end{aligned}$$

¹Compton scattering in an inhomogeneous medium will produce some polarization of the photons, which can be measured, and also effects the anisotropy at the several percent level. See Melchiorri and Vittorio this volume.

²This is true for a phase space defined by a position, x^i , and its canonically conjugate momentum, p_i , $n(\mathbf{p})$ measures the particle density with volume measure: $d^3x^i d^3p_i$. In general relativity there is both the covariant momentum, p_i , and contravariant momentum, p^i . If one measures the density of particles per unit $d^3x^i d^3p^i$ the Boltzmann equation as expressed above does not apply!

³This form is determined by the *principle of detailed balance* which results from the time-reversal symmetry of the S-matrix (or classical or quantum mechanics)[45].

where we have (or will) use the notation

$$\begin{aligned} \mathbf{p}_\gamma &= \frac{\epsilon}{c} \hat{\mathbf{n}} & \mathbf{p}'_\gamma &= \frac{\epsilon'}{c} \hat{\mathbf{n}}' & |\hat{\mathbf{n}}| &= |\hat{\mathbf{n}}'| = 1 & E &= \sqrt{(m_e c^2)^2 + (c \mathbf{p}_e)^2} \\ \mathbf{p}_e &= (m_e c) \gamma \vec{\beta} & \beta &= |\vec{\beta}| & \gamma &= \frac{1}{\sqrt{1 - \beta^2}}. \end{aligned} \quad (17)$$

In eq 16 the values of \mathbf{p}'_e and ϵ' is determined by energy-momentum conservation. The 2nd term in square brackets describes the scattering $\mathbf{p}_\gamma + \mathbf{p}_e \rightarrow \mathbf{p}'_\gamma + \mathbf{p}'_e$ while the 1st term results from the inverse process, $\mathbf{p}'_\gamma + \mathbf{p}'_e \rightarrow \mathbf{p}_\gamma + \mathbf{p}_e$. The $1 + n_\gamma$ factor represents the increased scattering rate due to the stimulated emission of the bosonic photons, while the $1 - n_\gamma$ is the Pauli blocking factor giving the exclusion principle for fermionic electrons. The factor of 2 in the prefactor counts the two polarization states of the incoming electrons. The factor $c(1 - \hat{\mathbf{n}} \cdot \vec{\beta})$ in eq 16 is a measure of the relative velocity between the ingoing electron and photon.⁴ Of course, $\frac{d^2 \sigma}{d^2 \hat{\mathbf{n}}}$ is the differential Compton cross-section⁵

Note that this form of the collision integral guarantees that a thermal distribution is a fixed point. Substituting a Fermi-Dirac distribution for the electrons and a Bose-Einstein distribution for the photon, i.e.

$$n_e(E) = \frac{1}{\exp(\frac{E}{k_B T} + \mu_e) + 1} \quad n_\gamma(\epsilon) = \frac{1}{\exp(\frac{\epsilon}{k_B T} + \mu_\gamma) + 1} \quad (18)$$

will cause the integrand of the collision integral to zero so long as the temperature is same for both. Here μ_e , and μ_γ are (dimensionless) chemical potentials given by the total electron and photon density, each of which is conserved by Compton scattering. We expect such a thermal distribution to be a stable fixed point since it is the highest entropy state and entropy increases according to Boltzmann's H-theorem[45]. In the contexts we are interested in the density of electrons is sufficiently low that $\mu_e \gg 1$ and Fermi-blocking is unimportant so we may set $1 - n_e \rightarrow 1$. In this limit the equilibrium distribution for the electrons becomes a simple Boltzmann distribution, i.e.,

$$n_e(E) = \exp(-\frac{E}{k_B T} - \mu_e). \quad (19)$$

⁴This relative velocity factor is really determined by the definition of the cross-section. The factor is equal to $\sqrt{|\mathbf{v}_1 - \mathbf{v}_2|^2 - \frac{1}{c^2} |\mathbf{v}_1 \times \mathbf{v}_2|^2}$ which reduces to the above expression when one of the particles is massless. If both incoming particles are non-relativistic then it reduces to the “usual” definition of relative-velocity: $|\mathbf{v}_1 - \mathbf{v}_2|$.

⁵The outgoing particle momentum \mathbf{p}_e and \mathbf{p}_γ are described by six numbers however four are fixed by energy and momentum conservation. The differential cross-section is a function of the remaining two parameters, which in this case we have taken to be the outgoing photon direction, $\hat{\mathbf{n}}'$. Any two parameters would do!

Henceforth we will ignore Fermi-blocking.

We are not really interested in the scattered electrons, so we may “integrate out” the electron distribution function. The idea is that we know the electron distribution function *a priori* - which is often is a true since Coulomb scattering is usually very effective in thermalizing the electron momenta. Thus we may rewrite the collision integral as

$$C^C(\mathbf{p}_\gamma) = \int d^2\hat{\mathbf{p}}'_\gamma \left[\frac{\epsilon^2}{\epsilon'^2} S(\mathbf{p}'_\gamma, \mathbf{p}_\gamma) (1 + n_\gamma(\mathbf{p}_\gamma)) n_\gamma(\mathbf{p}'_\gamma) - S(\mathbf{p}_\gamma, \mathbf{p}'_\gamma) (1 + n_\gamma(\mathbf{p}'_\gamma)) n_\gamma(\mathbf{p}_\gamma) \right] \quad (20)$$

where

$$S(\mathbf{p}_\gamma, \mathbf{p}'_\gamma) = \frac{2}{(2\pi\hbar)^3} \int d^3\mathbf{p}_e n_e(\mathbf{p}_e) (1 - \vec{\beta} \cdot \hat{\mathbf{n}}) \frac{d^2\sigma}{d^2\hat{\mathbf{n}}'}(\mathbf{p}_e, \mathbf{p}_\gamma, \hat{\mathbf{n}}') \frac{\delta(\epsilon' - \epsilon(1 + \bar{\Delta}))}{\epsilon'^2} \quad (21)$$

and $\bar{\Delta}(\mathbf{p}_e, \mathbf{p}_\gamma, \hat{\mathbf{n}}')$ gives the fractional change in the energy determined by energy-momentum conservation, i.e. is the solution to the equation

$$\begin{aligned} & \sqrt{(m_e c^2)^2 + |c\mathbf{p}_e|^2 + c|\mathbf{p}_\gamma|} \\ &= \sqrt{(m_e c^2)^2 + |c\mathbf{p}_e + c\mathbf{p}_\gamma - \epsilon(1 + \bar{\Delta})\hat{\mathbf{n}}'|^2 + c|\mathbf{p}_e|(1 + \bar{\Delta})}. \end{aligned} \quad (22)$$

A unique solution always exists with $\bar{\Delta} \in [-1, \infty)$.

We know that the CMBR is very nearly isotropic today, and it is reasonable to assume that the background radiation was always isotropic. Since we are interested in changes in the spectrum and not anisotropy we may also average the collision integral over $\hat{\mathbf{n}}$ to find the mean change in the spectrum. Performing the two averages $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ the collision integral becomes

$$C^C(\epsilon, \Delta) = \int d\Delta \left[\frac{1}{(1 + \Delta)^3} \bar{S}\left(\frac{\epsilon}{1 + \Delta}, \Delta\right) (1 + n_\gamma(\epsilon)) n_\gamma\left(\frac{\epsilon}{1 + \Delta}\right) - \bar{S}(\epsilon, \Delta) (1 + n_\gamma(\epsilon(1 + \Delta))) n_\gamma(\epsilon) \right]. \quad (23)$$

where

$$\bar{S}(\epsilon, \Delta) = \frac{\epsilon^3(1 + \Delta)^2}{4\pi} \int d^2\hat{\mathbf{n}} \int d^2\hat{\mathbf{n}}' S\left(\frac{\epsilon}{c}\hat{\mathbf{n}}, \frac{\epsilon}{c}(1 + \Delta)\hat{\mathbf{n}}'\right). \quad (24)$$

To obtain eq 23 we have used a little trick of changing the variable of integration for inverse scattering from Δ to $\frac{1}{1 + \Delta} - 1$, and renaming this

new dummy variable Δ . If one looks closely at eq 23 one can see that the total photon number is preserved by scattering no matter what the form of $\bar{S}(\epsilon, \Delta)$.

6.1.2. Fokker-Planck Equation

One important property of cosmological Compton scattering is that, at the low redshifts we are interested in, the background radiation photons have much lower (total) energy and are moving much faster than the electron they are scattering off of. One is bouncing a very light object (the photon) off of a much more slowly moving heavy object (the electron) and energy and momentum conservation dictates that that energy of the light object is nearly unchanged by the scattering (consider bouncing a ping-pong ball off of a bowling ball). The electrons are not infinitely massive nor are they completely stationary so that the photon energy will change slightly in each collision. All this will be reflected in the fact that $\bar{S}(\epsilon, \Delta)$ when considered as a function of Δ will be a very narrow function sharply peaked around $\Delta = 0$ with width much less than unity. In contrast the Δ -dependence of $n_\gamma(\epsilon(1 + \Delta))$ and $\bar{S}(\frac{\epsilon}{1+\Delta},)$ is a much smoother function in the sense that they do not vary much over the region in Δ where $\bar{S}(\epsilon, \Delta)$ is significantly non-zero. Thus it should be a good approximation to Taylor expand the integrand of eq 23 in Δ about $\Delta = 0$, but excluding the rapid dependence through the 2nd argument of \bar{S} and truncating at a given order. This is a kind of Fokker-Planck equation⁶. If we expand to 2nd order in Δ the Boltzmann equation becomes a partial differential equation (see eq 8 of ref [1])

$$\frac{Dn_\gamma}{D\tau} = \frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon} \left[\epsilon^3 \left(\frac{1}{2} \epsilon \overline{\Delta^2} \frac{\partial n_\gamma}{\partial \epsilon} + \left(-\overline{\Delta} + 2\overline{\Delta^2} + \frac{1}{2} \epsilon \overline{\Delta^{2'}} \right) (1 + n_\gamma) n_\gamma \right) \right] \quad (25)$$

where

$$\overline{\Delta^n} = \frac{1}{N_e \sigma_T} \int_{-1}^{\infty} d\Delta \Delta^n \bar{S}(\epsilon, \Delta) \quad \overline{\Delta^{n'}} = \frac{1}{N_e \sigma_T} \int_{-1}^{\infty} d\Delta \Delta^n \frac{\partial \bar{S}(\epsilon, \Delta)}{\partial \epsilon} \quad (26)$$

⁶Fokker and Planck actually considered the case where the momentum is only slightly changed in each scattering and proposed Taylor expanding to 2nd order in the small change in momentum. For Compton scattering the direction of the photon will change significantly so the momentum change is not small, but the energy change is, and expanding in the small fractional energy change, Δ , is an obvious generalization. It is useful to consider expanding to higher order than 2nd.

and we have used the electron density, N_e , introduced the Thomson cross-section⁷, σ_T , and defined the Thomson optical depth, τ :

$$N_e = \frac{2}{(2\pi\hbar)^3} \int d^3\mathbf{p}_e n_e(\mathbf{p}_e) \quad \sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 \quad d\tau = N_e c \sigma_T dt . \quad (27)$$

This optical depth gives the expected number of Compton scatterings of low energy photons off of non-relativistic electrons.

The form of the equation is reminiscent of a diffusion equation which is good description of the physics, the small changes in photon energy at each scattering causes the photons to diffuse in energy space. The $\frac{\partial n_\gamma}{\partial \epsilon}$ term causes a net drift toward increasing energies while the $(1 + n_\gamma)n_\gamma$ will cause a net drift to lower energies (if it's coefficient is positive). We expect these drifts and diffusion to sum to zero in thermal equilibrium, i.e. when n_γ has a Bose-Einstein distribution (eq 18), the electrons have a Boltzmann distribution (eq 19), and the two share a common temperature. This consideration alone suggest that for a thermal electron distribution with temperature T_e that we should expect

$$\frac{-2\overline{\Delta} + 4\overline{\Delta^2} + \epsilon \overline{\Delta^{2'}}}{\overline{\Delta^2}} = \frac{\epsilon}{kT_e} . \quad (28)$$

Another feature of eq 25 is the differential operator $\frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon}$ in front, which guarantees conservation of photon number. This will persist to all order in the Δ -expansion. In fact one can pretty much guess the 2nd order Fokker-Planck equation without knowing much about the Compton cross-section. We will take a more constructive approach below.

6.1.3. Compton Cross-Section

To compute the Compton collision integral, or it's Fokker-Planck approximations one needs to use the Compton cross-section. The relativistic (Klein-Nishina) differential Compton cross-section in an arbitrary rest-frame is[1]

$$\begin{aligned} \frac{d^2\sigma}{d^2\hat{\mathbf{n}}'} = \frac{3\sigma_T}{16\pi} & \frac{1 - \beta^2}{[1 - \hat{\mathbf{n}}' \cdot \vec{\beta} + \alpha\gamma^{-1}(1 - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')]^2} \\ & \times \left[1 + \left(1 - \frac{(1 - \beta^2)(1 - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')}{(1 - \hat{\mathbf{n}} \cdot \vec{\beta})(1 - \hat{\mathbf{n}}' \cdot \vec{\beta})} \right)^2 \right. \\ & \left. + \frac{\alpha^2(1 - \beta^2)(1 - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')^2}{(1 - \hat{\mathbf{n}}' \cdot \vec{\beta})[1 - \hat{\mathbf{n}}' \cdot \vec{\beta} + \alpha\gamma^{-1}(1 - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')]} \right] \quad (29) \end{aligned}$$

⁷Thomson scattering is the non-relativistic and classical limit of Compton scattering as first described by J.J. Thomson.

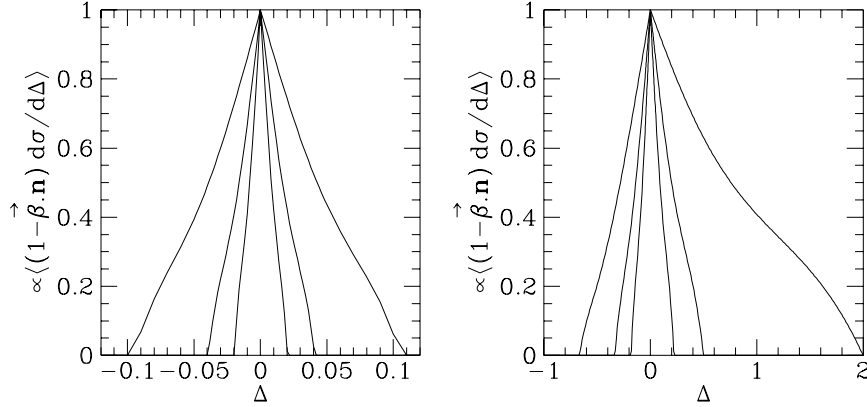


Figure 8. Plotted is the distribution of fractional energy changes, Δ , experienced by low energy photons scattering off of an isotropic distribution of electrons with velocity βc . The left panel shows the distribution for $\beta = 0.01, 0.02$, and 0.05 ; while the right panel shows the distribution for $\beta = 0.1, 0.2$, and 0.5 . For graphical clarity we have adjusted the heights of the curves to have unit amplitude at $\Delta = 0$. The maximum and minimum values for Δ are dictated by energy and momentum conservation. The distribution is narrow and symmetric for small β and becomes wider and more skew for larger velocity electrons. This positive skewness gives the heating of the photons by the electrons. The Fokker-Planck equation approximates the photon distribution function by the first few terms in it's Taylor series about $\Delta = 0$ when convolving with these distributions. This is liable to be a good approximation for scattering off of low velocity electrons since the Δ -distribution is sharply peaked around $\Delta = 0$.

where

$$\alpha = \frac{\epsilon}{m_e c^2} . \quad (30)$$

For many astrophysical applications, and especially those related to the CMBR there are two small numbers which enter this cross-section. Firstly α is very small for the microwave photons we see observe today, roughly 10^{-9} . Clearly as we go to higher redshifts the background photons become more energetic, but α remains small in the redshift range relevant to the CMBR spectrum $z \lesssim 10^7$. The 2nd small number is β since we are almost always interested in non-relativistic electrons. If one is interested in a thermal electron velocity distribution then a small β expansion is equivalent to a small $\frac{kT_e}{m_e c^2}$ expansion. In most applications the $\alpha \ll \frac{kT_e}{m_e c^2}$ so we will concentrate more on the higher order terms in T_e and not α .

To proceed it is probably easiest to follow the methodology of Barbosa [1], where one expands the cross-section in α but not β . For a thermal electron distribution one can compute the moments, $\overline{\overline{\Delta^n}}$, in the Fokker-Planck

expansion analytically, and only at the end one should Taylor expand the result in T_e about $T_e = 0$. One may rewrite eq 24 as

$$\overline{S}(\epsilon, \Delta) = N_e \left\langle (1 - \vec{\beta} \cdot \hat{\mathbf{n}}) \frac{d\sigma}{d\Delta} \right\rangle_{\hat{\mathbf{n}}, \hat{\mathbf{n}}'} \quad (31)$$

where $\frac{d\sigma}{d\Delta}$ gives the differential cross-section wrt to the fractional change in photon energy. So for example, expanding everything to zeroth order in α (which we denote by the superscript $^{(0)}$) we find

$$\left\langle (1 - \vec{\beta} \cdot \hat{\mathbf{n}}) \frac{d\sigma^{(0)}}{d\Delta} \right\rangle_{\hat{\mathbf{n}}, \hat{\mathbf{n}}'} = \sigma_T \overline{F}(\Delta, \beta \operatorname{sgn}(\Delta)) \quad (32)$$

where

$$\begin{aligned} \overline{F}(\Delta, b) = \operatorname{sgn}(\Delta) \times & \mathcal{H}\left(1 - \frac{(1-b)\Delta}{2b}\right) \\ & \times \left[\frac{3(1-b^2)^2(3-b^2)(2+\Delta)}{16b^6} \ln \frac{(1-b)(1+\Delta)}{1+b} \right. \\ & + \frac{3(1-b^2)(2b - (1-b)\Delta)}{32b^6(1+\Delta)} \left(4(3-3b^2+b^4) \right. \\ & \quad \left. + 2(6+b-6b^2-b^3+2b^4)\Delta \right. \\ & \quad \left. + (1-b^2)(1+b)\Delta^2 \right) \left. \right], \quad (33) \end{aligned}$$

and $\mathcal{H}()$ is the Lorentz-Heaviside function which is unity for positive argument and zero otherwise. We see that this function is only non-zero for

$$\Delta \in \left[-\frac{2\beta}{1+\beta}, \frac{2\beta}{1-\beta} \right] \quad (34)$$

and, as promised, for small β is sharply peaked around $\Delta = 0$. We plot this function for various values of β in fig 8.

6.1.4. Moments of Δ

With this general expression for $\frac{d\sigma^{(0)}}{d\Delta}$ given above one can compute, to 0th order in α , the Δ -moments which are the coefficients in the Fokker-Planck equation (some of these may be found in ref [1]:

$$\begin{aligned} \overline{\overline{\Delta^0}}^{(0)} &= 1 \\ \overline{\overline{\Delta^1}}^{(0)} &= \frac{4}{3} \gamma^2 \beta^2 = 4 \left(\frac{kT_e}{m_e c^2} \right) + 10 \left(\frac{kT_e}{m_e c^2} \right)^2 + \mathcal{O} \left[\left(\frac{kT_e}{m_e c^2} \right)^3 \right] \end{aligned}$$

$$\begin{aligned}
\overline{\overline{\Delta^2}}^{(0)} &= \frac{2}{15} \overline{\gamma^4 \beta^2 (5 + 16\beta^2)} = 2 \left(\frac{kT_e}{m_e c^2} \right) + 47 \left(\frac{kT_e}{m_e c^2} \right)^2 + \mathcal{O} \left[\left(\frac{kT_e}{m_e c^2} \right)^3 \right] \\
\overline{\overline{\Delta^3}}^{(0)} &= \frac{4}{25} \overline{\gamma^6 \beta^4 (21 + 23\beta^2)} = \frac{252}{5} \left(\frac{kT_e}{m_e c^2} \right)^2 + \mathcal{O} \left[\left(\frac{kT_e}{m_e c^2} \right)^3 \right] \\
\overline{\overline{\Delta^4}}^{(0)} &= \frac{4}{525} \overline{\gamma^8 \beta^4 (147 + 1554\beta^2 + 859\beta^4)} = \frac{84}{5} \left(\frac{kT_e}{m_e c^2} \right)^2 + \mathcal{O} \left[\left(\frac{kT_e}{m_e c^2} \right)^3 \right].
\end{aligned} \tag{35}$$

and we also find that $\overline{\overline{\Delta^{n'}}}^{(0)} = 0$ since $\frac{d\sigma^{(0)}}{d\Delta}$ has no dependence on ϵ . The fact that $\overline{\overline{\Delta^0}}^{(0)} = 1$ tells us that, to 0th order in α and all orders in β the scattering rate per unit volume is $cN_e\sigma_T$.⁸ The coefficients in the Fokker-Planck equations are determined by the average of the electron velocities indicated, and these expressions hold whether or not the electrons are in thermal equilibrium. For a thermal distribution these velocity moments can be computed exactly in terms of modified Bessel functions[1], however we have found it convenient to expand these functions to the appropriate order in temperature. It seems that a Taylor series to a given order in Δ is less accurate than the same order Taylor series expansion in T_e . To keep track of the various terms in the expansion let us devise the notation

$$\mathcal{O}(n, m) = \mathcal{O} \left(\left(\frac{kT_e}{m_e c^2} \right)^n \left(\frac{\epsilon}{m_e c^2} \right)^m \right) \tag{36}$$

There are no terms $\sim \mathcal{O}(0, 0)$. One finds that

$$\overline{\overline{\Delta^{2n-1}}}^{(m)} \sim \overline{\overline{\Delta^{2n}}}^{(m)} \sim \mathcal{O}(n, m). \tag{37}$$

so to include all the terms of order $\sim \mathcal{O}(n, m)$ in one must make a Fokker-Planck expansion to order $2n$ in Δ . It is probably not worthwhile to go to high order in these expansions, since one can circumvent this expansion by doing the collision integral. Nevertheless the first few terms give useful analytical expressions.

⁸The total (Klein-Nishina) cross-section starts to fall below the Thomson cross-section when the center-of-mass photon energy rises to close to $m_e c^2$, i.e. when $\gamma\alpha \gtrsim 1$. A careful look at eq 29 will show that by setting $\alpha = 0$ in this equation we are ignoring terms of order $\alpha\gamma$. For microwave photons this approximation should be good for computing the total cross-section as long as $\gamma \lesssim 10^9$ i.e. for anything less energetic than $\sim 500\text{TeV}$ electrons. In contrast to compute the small effects on the spectrum from Compton scattering one should include 1st order terms in α whenever $\alpha \gtrsim \beta^2$.

6.1.5. The Kompaneets Equation and Relativistic Corrections

The lowest order non-zero Fokker-Planck equation, given by the expansion of eq 25, is the Kompaneets equation

$$\frac{\partial n_\gamma}{\partial \tau} = \frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon} \left[\epsilon^3 \left(\underbrace{\frac{kT_e}{m_e c^2} \epsilon \frac{\partial n_\gamma}{\partial \epsilon}}_{\mathcal{O}(1,0)} + \underbrace{\frac{\epsilon}{m_e c^2} (1 + n_\gamma) n_\gamma}_{\mathcal{O}(0,1)} \right) \right] \quad (38)$$

where the order of the two terms are indicated. This equation was first published by Kompaneets[42] in 1957 and probably developed earlier as part of the Soviet thermonuclear weapons program. For hotter gas one can add terms $\mathcal{O}(2,0)$ which yields an extended Kompaneets equation[72]

$$\begin{aligned} \frac{\partial n_\gamma}{\partial \tau} = \frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon} \left[\epsilon^3 \left(\frac{kT_e}{m_e c^2} \left(1 + \frac{5}{2} \frac{kT_e}{m_e c^2} \right) \epsilon \frac{\partial n_\gamma}{\partial \epsilon} \right. \right. \\ \left. \left. + \frac{7}{10} \left(\frac{kT_e}{m_e c^2} \right)^2 \left(6\epsilon^2 \frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \epsilon^3 \frac{\partial^3 n_\gamma}{\partial \epsilon^3} \right) \right. \right. \\ \left. \left. + \frac{\epsilon}{m_e c^2} (1 + n_\gamma) n_\gamma \right) \right] \quad (39) \end{aligned}$$

Further terms in this expansion will be derived in ref [72] although it is not clear how useful they will be since extensive numerical work has been done with the more accurate collision integral (e.g. ref [56]).

6.1.6. The Generalized Sunyaev-Zel'dovich Effect

The idea of the Sunyaev Zel'dovich distortion is that one starts out with a background radiation which is close to a blackbody spectrum, just what we expect to be produced by the early universe, and it is *slightly* distorted by the action of hot ionized gas through the Compton scattering process we have just described. In this small distortion limit we need just substitute in a blackbody spectrum, n^{BB} of eq 2, into the right-hand-side of the Kompaneets equation. Let us generalize this idea a bit by instead considering the more general Fokker-Planck expansion which is an expansion in T_e and α . In this small distortion limit the different terms will add linearly to the total distortion which we may write as a sum

$$\Delta n_\gamma = \sum_{n \geq 0} \sum_{m \geq 0} Y_C^{(n,m)} \Delta n_{\text{SZ}}^{(n,m)}(x) \quad x = \frac{\epsilon}{kT_\gamma} \quad (40)$$

where

$$Y_C^{(n,m)} = \int d\tau \left(\frac{kT_e}{m_e c^2} \right)^n \left(\frac{kT_\gamma}{m_e c^2} \right)^m \quad (41)$$

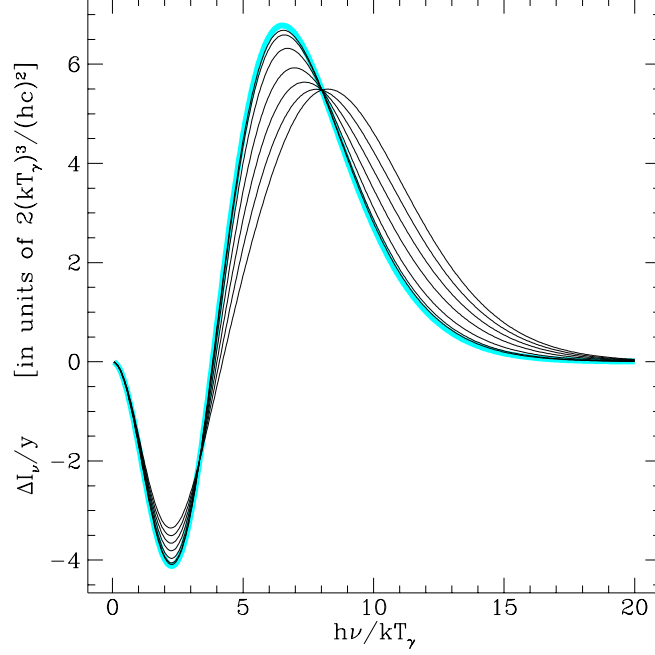


Figure 9. Plotted is the small deviation in intensity from a blackbody divided by the classical y -parameter caused when blackbody photons pass through a hot gas of electrons. This is computed using the extension of the y -distortion given in the text. The gray band is centered on the classical y -distortion which applies when $kT_e \ll m_e c^2$. The black lines are for electron temperatures of 1, 2, 5, 10, 15, 20, and 25 keV. We have of course assumed $T_e \gg T_\gamma$. The curves intersect at the zeros of the function $\Delta n_{SZ}^{(2,0)}$.

and the superscript $^{(n,m)}$ correspond to the $\mathcal{O}(n,m)$ contributions to the Fokker-Planck expansion. Substituting $n^{\text{BB}}(x)$ into the various terms of eq 39 we find that

$$\begin{aligned}
 \Delta n_{SZ}^{(0,0)}(x) &= 0 \\
 \Delta n_{SZ}^{(1,0)}(x) &= \frac{x e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) \\
 \Delta n_{SZ}^{(0,1)}(x) &= -\frac{x e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) \\
 \Delta n_{SZ}^{(2,0)}(x) &= \frac{x e^x}{(e^x - 1)^2} \left(-10 + \frac{47}{2} x \frac{e^x + 1}{e^x - 1} - \frac{42}{5} x^2 \frac{e^{2x} + 4e^x + 1}{(e^x - 1)^2} \right. \\
 &\quad \left. + \frac{7}{10} x^3 \frac{(e^x + 1)(e^{2x} + 10e^x + 1)}{(e^x - 1)^3} \right). \quad (42)
 \end{aligned}$$

One does expect that to each order in energy that a blackbody spectrum is a stable solution when the electron and photon temperature are equal so one should expect the sum rule

$$\sum_{n=0}^N \Delta n_{\text{SZ}}^{(n, N-n)}(x) = 0 \quad (43)$$

and this does seem to be true for $N = 0$ and $N = 1$.

The classical Sunyaev-Zel'dovich y -distortion contains only the $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ terms and may be written

$$\Delta n = y \frac{x e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) \quad (44)$$

where

$$y = Y_{\text{C}}^{(1,0)} - Y_{\text{C}}^{(0,1)} = \int d\tau \frac{k(T_{\text{e}} - T_{\gamma})}{m_{\text{e}} c^2}. \quad (45)$$

This is the y -distortion plotted in fig 5 and used in eqs 11&12. To see how much this classical formula errs we compare the different expression for a range of electron temperature in fig 9. We see that the $\mathcal{O}(2, 0)$ corrections become significant when $T_{\text{e}} \gtrsim 5\text{keV}$. This 2nd order distortion agrees well with the computation of the collision integral by Rephaeli[56] so higher order corrections do not seem to be important for $T_{\text{e}} \lesssim 15\text{keV}$.

7. The Future

In the past decade we have witnessed astounding advances in the measurement of the CMBR spectrum. After decades of tantalizing evidence of deviations from a blackbody spectrum we find that the spectrum is amazingly close to a perfect blackbody. No longer is it possible to consider a universe with a very hot inter-galactic medium, or that hydrodynamic forces could have played a large role in forming the *large* scale structure. There is also little room for non-equilibrium energetic events in the early universe at redshifts $< 10^7$. In a way this is most unfortunate. The thermal equilibrium state contains the least information - all remnant of events in the universe before $z \sim 10^7$ have been thermalized to nothing, or more precisely to one number: the temperature. At the moment we really don't know how to interpret this number, other than to make a rough comparison to the number of baryons which is observationally rather ill-determined. Perhaps some day we will have cosmogenic theories which will predict the baryon-to-photon ratio with great accuracy.

Observationally we are approaching a brick wall which is the Galaxy. At the present level of sensitivity Galactic contamination from dust and synchrotron radiation is an important contaminant at all wavelengths. Galaxy

modeling which makes use of a spectral and spatial structure of the Galaxy observed at a variety of wavelengths will improve as sensitivities improve however there will be a limit to how accurately one can subtract off the Galaxy even given perfect data. We won't be making observations outside of the Galactic plane any time soon!

Yet there is still a lot of room for improvement on the decimeter and meter scale anisotropies. Also there is this tantalizing evidence for *negative* spectral deviations⁹.

Things are not bleak. In fact spectral distortions of the CMBR is a rapidly growing field. Multi-frequency observations is beginning to be the norm for CMBR anisotropy experiments, and with the CMBR satellites we can expect literally millions of measurements of the CMBR spectrum in different directions on the sky. Admittedly there is a big difference between absolute measurements of the CMBR flux and differential measurements which are required for anisotropy since the anisotropy spectral measurements are modulo any DC spectral distortion. However it is just this sort of measurement which will make improved Galaxy subtraction possible. The spectral information obtained will tell us mostly about the Galaxy and extra-Galactic radio sources, however with millions of measurements one can always hope for something a little more interesting. Along these lines there is the cluster S-Z effect which is a rapidly growing field. With increased sensitivity we can look forward to S-Z selected cluster catalogs, measurements of radial cluster velocities through the kinematic S-Z effect, and these studies can work their way down to galaxy groups and even large scale structure filaments of hot gas. We can even hope to measure the gas temperature from spectrum if it is hot enough. In the future we can expect the spectrum and anisotropy measurements to become increasingly intertwined.

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9. Bibliography

What follows is not a list of articles cited in this work, although it includes all articles cited, but rather an (incomplete) bibliography of published works related to the CMBR spectrum, including title, listed alphabetically by the name of the first author. Many of these papers are of only historic interest:

⁹Some people never learn.

theories have been ruled out and observations superseded. I hope some readers will find it a useful reference.¹⁰

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¹⁰The compilation method was somewhat haphazard and the author apologizes to the authors of the many important works which are not listed. Incompleteness is probably fairly large for papers written in the past decade, and entire subject areas (e.g. decaying particles) have been omitted. Conference proceedings and other articles in books have been excluded.

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